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Abstract: In this paper, the problem of state estimation is investigated for linear switched system, a subclass of hybrid systems. It will be shown that the interval observer is very often exists under moderate conditions at least in discrete time instants from continuous-time measurements. The novelty consists in proposing new conditions of cooperativity for switched systems in discrete time instants, which guarantee errors nonnegativity of interval observation. The efficiency of the interval observers is shown through simulation examples.

Keywords: Nonnegative dynamics, Interval observer, Switched system.

1. INTRODUCTION

Switched systems are very flexible modeling tools, which appear in several fields such as networked control systems, electrical devices/circuits, congestion modeling (Liberzon, 2003; Briat, 2015). This class of systems is viewed as an abstraction of hybrid systems, obtained by neglecting the discrete dynamics (Lin and Antsaklis, 2009; Liberzon et al., 2004). State estimation of switched systems has received considerable attention over past decades. In the case of switched system with state jumps, necessary and sufficient conditions for the observability have been established based on graph-theoretic approach (Boukhobza and Hamelin, 2011), and geometric approach (Tanwani et al., 2013). In (Barbot et al., 2007; Ríos et al., 2015) sliding mode observers have been designed to estimate the continuous and discrete states for the switched system in observability canonical form.

The estimation problem becomes much more involved if we consider systems which are subjected to model and/or signals uncertainties. Therefore, the state estimation approaches based on the set-membership and interval observers get more attention for uncertain systems (Efimov and Raïssi, 2016). In the literature, the interval observers are applied to estimate the state for several classes of nonlinear systems (Raissi et al., 2012; Efimov et al., 2013c), time-delay systems (Efimov et al., 2013b) and sampled-data systems (Mazenc et al., 2014; Efimov et al., 2013a). Several interval observers for impulsive systems, a class of linear hybrid systems, have been recently applied in (Degue et al., 2016).

For switched systems the switching among subsystems can affect the cooperativity property of interval estimation. In this work, we deal with the synthesis of interval estimators for switched system under bounded unknown input, and a new condition of cooperativity is proposed. The stability of the interval observer is guaranteed by using Lyapunov stability analysis based on Common Lyapunov Function (CLF) that decreases independently of switching (Liberzon et al., 2004).

⋆ This work was partially supported by the Government of Russian Federation (Grant 074-U01) and the Ministry of Education and Science of Russian Federation (Project 14.Z50.31.0031).

2. PRELIMINARIES

Euclidean norm for a vector $x \in \mathbb{R}^n$ will be denoted as $|x|_e$, and for a measurable and locally essentially bounded input $u : \mathbb{R}_+ \to \mathbb{R}$ ($\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$) the symbol $||u||_{[t_0,t_1]}$ denotes its L_∞ norm:

$$||u||_{[t_0,t_1]} = ess\ sup\{|u(t)|_e, t \in [t_0, t_1]\},$$

if $t_1 = +\infty$ then we will simply write $||u||$. By $\mathcal{L}_\infty$ we will denote the set of all inputs u with the property $||u|| < \infty$. By $\overline{1,k}$ we denote the sequence of integers $1, ..., k$. Any $p \times m$ matrix whose elements are all ones or zeros are simply denoted by $E_{p,m}$ or 0, respectively. I_p denotes the identity matrix in $\mathbb{R}^{p\times p}$.
Throughout this paper the inequalities must be understood component-wise, for matrices as well as for vectors, i.e. $A = (a_{i,j}) \in \mathbb{R}^{p\times m}$ and $B = (b_{i,j}) \in \mathbb{R}^{p\times m}$ such that $A \geq B$ if and only if, $a_{i,j} \geq b_{i,j}$ for all $i \in \{1, \ldots, p\}, j \in \{1, \ldots, m\}$. $M = \max\{A, B\}$ is the matrix where each entry is $m_{i,j} = \max\{a_{i,j}, b_{i,j}\}$. Let us define $A^+ = \max\{A, 0\}$, $A^- = A^+ - A$; thus, the element-wise absolute value will be denoted as $|A| = A^+ + A^-$. A matrix $P \in \mathbb{R}^{n\times n}$ is said to be negative definite if $v^T P v < 0$ for all non-zero real vectors $v \in \mathbb{R}^n$ and it will be denoted by $P \prec 0$. Similarly, $P \preceq 0 (P \succ 0)$ means semi-negative (positive) definite matrix.

2.1 Cooperativity

Definition 1. (Minc, 1988) A matrix $M = (m_{i,j}) \in \mathbb{R}^{n\times n}$ is said to be Metzler if all its off-diagonal elements are nonnegative i.e. $m_{i,j} \geq 0,\ \forall(i, j),\ i \neq j$. And it is said to be Nonnegative if every entries are nonnegative: $M \geq 0$.

Definition 2. (Farina and Rinaldi, 2000) A continuous-time linear system $\dot{x}(t) = Ax(t)$ (discrete-time linear system $x(t+1) = Ax(t)$), with the state $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, is said to be cooperative if A is a Metzler (Nonnegative) matrix.

The solutions of cooperative systems, initiated from $x(0) \geq 0$, stay nonnegative: $x(t) \geq 0$ for all $t \geq 0$.

2.2 Intervals

Lemma 1. (Efimov et al., 2012) Let $A \in \mathbb{R}^{p\times m}$ be a constant matrix and the vector $x \in \mathbb{R}^m$ be variable with some bounds $\underline{x}, \overline{x} \in \mathbb{R}^m$ such that $\underline{x} \leq x \leq \overline{x}$, then

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}.$$

2.3 Stability

A continuous-time switched linear time-invariant (LTI) system can be represented as

$$\dot{x}(t) = A_{\sigma(t)}x(t),\ \sigma(t) \in \mathcal{I} = \{1, \dots, N\} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, the finite set $\mathcal{I}$ is an index set and it stands for the collection of discrete modes $A_i \in \mathbb{R}^{n\times n}$ for all $i \in \mathcal{I}$. The logical rule that arranges the switching between the subsystems generates a switching signal, a piecewise constant function $\sigma(t) : \mathbb{R}_+ \to \mathcal{I}$, the index $i = \sigma(t)$ specifies, at each instant of time, the system that currently being followed.
By saying that the switching signal is piecewise constant we mean that it has a finite number of discontinuities on any finite interval of time. We assume that $\sigma(t)$ is continuous from the right everywhere: $\sigma(t) = \lim_{\alpha\to t^+} \sigma(\alpha)$.
We will say that the switching signals satisfy the so-called dwell-time property if $t_{k+1} - t_k \geq \tau_D$ for some dwell time constant $\tau_D > 0$, where t_k is the instant of k^{th} switch.

Lemma 2. (Liberzon et al., 2004) Let $P \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix that satisfies the Linear Matrix Inequalities (LMIs)

$$A_i^T P + PA_i \prec 0,\ i \in \mathcal{I} = \{1, \dots, N\} \tag{2}$$

Then $V(x) = x^T Px$ is a Common Quadratic Lyapunov Function (CQLF) for the systems (1).

This lemma establishes conditions of the internal stability (without taking into account the effect of external inputs). For switched systems with inputs and dwell-time switching signals, the overall system is input-to-state stable (ISS) if the individual subsystems are ISS (Vu et al., 2007). For linear switched systems with additive inputs, which we will consider below:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + d(t), \tag{3}$$

where $u(t) \in \mathbb{R}^m$, $u \in \mathcal{L}_\infty$ is a known input and $B_i \in \mathbb{R}^{n\times m}$ for all $i \in \mathcal{I}$, $d(t) \in \mathbb{R}^n$, $d \in \mathcal{L}_\infty$ is external disturbance, the conditions of Lemma 2 imply ISS.

3. MAIN RESULTS

Consider the switched linear system (3) with the output expression given by

$$y(t) = C_{\sigma(t)}x(t) + \nu(t) \tag{4}$$

where $y(t) \in \mathbb{R}^p$ is the output of the system, and $\nu(t)$ is the output measurement noise.
The models of subsystems can be represented as

$$\begin{aligned}\dot{x}(t) &= A_i x(t) + B_i u(t) + d(t) \quad i \in \mathcal{I}\\ &= (A_i - L_iC_i)x(t) + d(t) + B_iu(t) + L_iy(t) - L_i\nu(t)\end{aligned} \tag{5}$$

and $L_i \in \mathbb{R}^{n\times p}$ is an observer gain that will be defined later.

The goal here is to compute two bounds for the state of the considered system. To do so, several approaches are considered to satisfy the nonnegativity of the estimation error dynamics. A new cooperativity condition based on Trotter approximation is proposed.

3.1 Simple Interval Observer

In this section we will consider a structure of interval observer for (5) based on the classical Luenberger observer with proper assumptions that the estimation error dynamics are cooperative.

Assumption 1. There exist matrices gains L_i and a symmetric positive-definite matrix P such that $\forall i \in \mathcal{I}$:

(1) The LMIs
$$(A_i - L_iC_i)^T P + P(A_i - L_iC_i) \prec 0 \tag{6}$$
(2) The matrices $(A_i - L_iC_i)$ are Metzler.

If the Assumption 1.1 is satisfied, then $V(x(t)) = x^T Px$ is a CQLF for the system (5).

Assumption 2. Let two functions $\underline{d}, \overline{d} : \mathbb{R}_+ \to \mathbb{R}^n$, $\underline{d}, \overline{d} \in \mathcal{L}^n_\infty$ and the scalar constant $\mathcal{V} \geq 0$ be given such that

$$\underline{d}(t) \leq d(t) \leq \overline{d}(t),\ |\nu(t)| \leq \mathcal{V}E_p$$

are verified $\forall t \in \mathbb{R}_+$.

Under assumptions 1 and 2 the interval observer can be designed as

$$\begin{aligned}\dot{\underline{x}}(t) &= (A_i - L_iC_i)\underline{x}(t) + \underline{d}(t) + L_iy(t) + B_iu(t) - |L_i|\mathcal{V}E_p\\ \dot{\overline{x}}(t) &= (A_i - L_iC_i)\overline{x}(t) + \overline{d}(t) + L_iy(t) + B_iu(t) + |L_i|\mathcal{V}E_p\end{aligned} \tag{7}$$

where $|L_i| = L_i^+ + L_i^-$.

Theorem 1. Let Assumptions 1, 2 hold and $x \in \mathcal{L}^n_\infty$, then in (5) with the interval observer (7) the relations

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t)$$

are satisfied provided that $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$. In addition, $\overline{x}, \underline{x} \in \mathcal{L}^n_\infty$.

All proofs are omitted due to space limitations.

3.2 Relaxation of Cooperativity

The conditions of the cooperativity of the observer given in Assumption 1.2 are rather restrictive, and further we will consider several ways to relax them. The main approach deals with transformation of coordinates in order to obtain the estimation error in a cooperative form. A time-invariant interval observer can be designed by applying a change of coordinates proposed in (Raissi et al., 2012).
In this context the interval observer will be constructed

in the new state variables $z = S_i^{-1}x$, where S_i is a nonsingular matrix, selected such that the matrices $D_i = S_i^{-1}(A_i - L_iC_i)S_i$ are Metzler.

Applying the transformation on the system (5) we obtain

$$\dot{z}(t) = D_i z(t) + S_i^{-1}B_i u(t) + S_i^{-1}L_i y(t) + \delta_i(t) \qquad (8)$$

where $\delta_i(t) = S_i^{-1}(d(t) - L_i\nu(t))$.

However, it is worth noting that when we adopt the change of coordinates $x = S_i z,\ S_i \in \mathbb{R}^{n\times n}$, jumps in the new state may arise. This is due to the fact that, generally, the transformation matrices S_i assigned to subsystems are not the same. Therefore, after switching, the new state is

$$z(t_k) = \varphi_i z(t_k^-), \forall i \in \mathcal{I},$$

where matrix $\varphi_i = S_{i+1}^{-1}S_i$ is not identity matrix.

Consequently, instead of synthesis an interval observer for continuous state, the interval observer will be synthesised, in this new variables, for a switched system with state jumps, i.e. for a hybrid system. Thus, based on the expression of the output $y(t_{k+1}) = C_{i+1}S_i z(t_{k+1}^-) + \nu(t_{k+1})$, at the switching instant, the reset equation can be rewritten as follows

$$\begin{aligned} z(t_{k+1}) =& \big(\varphi_i - M_{i+1}C_{i+1}S_i\big) z(t_{k+1}^-) \\ &+ M_{i+1}y(t_{k+1}) - M_{i+1}\nu(t_{k+1}) \end{aligned} \qquad (9)$$

where $M_i \in \mathbb{R}^{n\times p}$ is another observer gain defined later. From (8), the interval observer is then given by

$$\begin{aligned} \underline{x}(t) =& S_i^+\underline{z}(t) - S_i^-\overline{z}(t), \\ \overline{x}(t) =& S_i^+\overline{z}(t) - S_i^-\underline{z}(t), \\ \dot{\underline{z}}(t) =& D_i\underline{z}(t) + S_i^{-1}(B_iu(t) + L_iy(t)) \\ &+ \underline{\delta}_i(t), \forall t \in [t_k, t_{k+1}), \forall k \in \mathbb{N}, \\ \underline{z}(t_{k+1}) =& \big(\varphi_i - M_{i+1}C_{i+1}S_i\big)\underline{z}_i(t_{k+1}^-) \\ &+ M_{i+1}y(t_{k+1}) - |M_{i+1}|\mathcal{V}E_p \\ \dot{\overline{z}}(t) =& D_i\overline{z}(t) + S_i^{-1}(B_iu(t) + L_iy(t)) \\ &+ \overline{\delta}_i(t), \forall t \in [t_k, t_{k+1}), \forall k \in \mathbb{N} \\ \overline{z}(t_{k+1}) =& \big(\varphi_i - M_{i+1}C_{i+1}S_i\big)\overline{z}_i(t_{k+1}^-) \\ &+ M_{i+1}y(t_{k+1}) + |M_{i+1}|\mathcal{V}E_p \end{aligned} \qquad (10)$$

where z is the state variable of (10), which is the same variable after and before jump, and

$$\begin{aligned} \underline{\delta}_i(t) &= (S_i^{-1})^+\Big[\underline{d}(t) - |L_i|\mathcal{V}E_p\Big] - (S_i^{-1})^-\Big[\overline{d}(t) + |L_i|\mathcal{V}E_p\Big], \\ \overline{\delta}_i(t) &= (S_i^{-1})^+\Big[\overline{d}(t) + |L_i|\mathcal{V}E_p\Big] - (S_i^{-1})^-\Big[\underline{d}(t) - |L_i|\mathcal{V}E_p\Big] \end{aligned}$$

with the initial conditions

$$\begin{aligned} \underline{z}(0) &= (S_i^{-1})^+\underline{x}(0) - (S_i^{-1})^-\overline{x}(0), \\ \overline{z}(0) &= (S_i^{-1})^+\overline{x}(0) - (S_i^{-1})^-\underline{x}(0). \end{aligned}$$

Assumption 3. There exist matrices $M_i,\ L_i \in \mathbb{R}^{n\times p}, i \in \mathcal{I}$ and a positive-definite matrix $P_\varphi \in \mathbb{R}^{n\times n}$ such that D_i are Metzler and $\big(\varphi_i - M_{i+1}C_{i+1}S_i\big)$ are nonnegative matrices, which satisfy the LMIs

$$\big(\varphi_i - M_{i+1}C_{i+1}S_i\big)^T e^{D_i^T\theta}P_\varphi e^{D_i\theta}\big(\varphi_i - M_{i+1}C_{i+1}S_i\big) - P_\varphi \prec 0 \qquad (11)$$

for all $\theta \in \big[\theta_{min}, \theta_{max}\big]$ and $0 \le \theta_{\min} \le t_{i+1} - t_i \le \theta_{\max} \le +\infty$ for all $i \ge 0$.

Note that in order to formulate stability conditions in the assumption above we need to introduce a kind of minimal $\theta_{\min}$ and maximal $\theta_{\max}$ dwell-time restrictions of the switching signal under consideration.

Theorem 2. Based on Assumptions 2 and 3, if the system (5) with some given $\underline{x}(0), \overline{x}(0) \in \mathbb{R}^n$ fulfills $\underline{x}(0) \le x(0) \le \overline{x}(0)$, then the established interval observer (10) satisfies the inclusion

$$\underline{x}(t) \le x(t) \le \overline{x}(t) \quad \forall t \ge 0$$

In addition, $\overline{x}, \underline{x} \in \mathcal{L}_\infty^n$.

3.3 An alternative of interval observer without coordinates change

An alternative solution to avoid the requirement that the matrices $(A_i - L_iC_i)$ for $i \in \mathcal{I}$ are Metzler has been proposed in (Cacace et al., 2015) (for linear systems), it is based on the fact that the system may be represented in its internally positive representation. However, this approach has more restrictive condition on stability. For the switched system (5) the state matrices can be presented as follows $(A_i - L_iC_i) = (A_i - L_iC_i)_d + (A_i - L_iC_i)_o^+ - (A_i - L_iC_i)_o^-$ with $i \in \mathcal{I}$ where $(\cdot)_d$ and $(\cdot)_o$ are the matrices that contain only the main diagonal elements and the off-diagonal elements, respectively. Thus the matrices $\big[(A_i - L_iC_i)_d + (A_i - L_iC_i)_o^+\big]$ for $i \in \mathcal{I}$ are Metzler. In the sequel, we shall denote for short $\tilde{A}_i = (A_i - L_iC_i)$.

Assumption 4. There exists a symmetric positive-definite matrix P_Υ that satisfies the LMIs

$$\Upsilon_i^T P_\Upsilon + P_\Upsilon \Upsilon_i \prec 0 \quad \forall i \in \mathcal{I} \qquad (12)$$

where

$$\Upsilon_i = \begin{pmatrix} \tilde{A}_{id} + \tilde{A}_{io}^+ & \tilde{A}_{io}^- \\ \tilde{A}_{io}^- & \tilde{A}_{id} + \tilde{A}_{io}^+ \end{pmatrix}.$$

Theorem 3. Let Assumptions 2, 4 be satisfied, $x \in \mathcal{L}_\infty^n$ in (5) and for some $\underline{x}(0), \overline{x}(0) \in \mathbb{R}^n$ the inclusion $\underline{x}(0) \le x(0) \le \overline{x}(0)$ is valid, then the interval observer

$$\begin{aligned} \dot{\underline{x}}(t) =& (\tilde{A}_{id} + \tilde{A}_{io}^+)\underline{x}(t) + \tilde{A}_{io}^-\overline{x}(t) + \underline{d}(t) \\ &+ L_iy(t) + B_iu(t) - |L_i|\mathcal{V}E_p \\ \dot{\overline{x}}(t) =& (\tilde{A}_{id} + \tilde{A}_{io}^+)\overline{x}(t) + \tilde{A}_{io}^-\underline{x}(t) + \overline{d}(t) \\ &+ L_iy(t) + B_iu(t) + |L_i|\mathcal{V}E_p \end{aligned} \qquad (13)$$

satisfies the order relations $\underline{x}(t) \le x(t) \le \overline{x}(t)$ and $\underline{x}(t), \overline{x}(t) \in \mathcal{L}_\infty^n$.

3.4 Trotter Formula Based Interval Observer

In this section we consider a special case when the gain matrices of the observer do not exist in a way to satisfy the cooperativity of the estimation error dynamics, neither by direct application nor by the means of coordinate transformation. In this case the following Trotter formula result can be used.

Theorem 4. (Trotter, 1959) For two given matrices $B, C \in \mathbb{R}^{n\times n}$, the exponentials of B and C are related to that of $B + C$ as follows

$$\lim_{m\to\infty}\Big(e^{\frac{B}{m}}e^{\frac{C}{m}}\Big)^m = e^{(B+C)}. \qquad (14)$$

The Trotter result can be used to evaluate e^A by splitting A into $B + C$ and then using the approximation

$$e^A \simeq \big(e^{\frac{B}{m}}e^{\frac{C}{m}}\big)^m$$

for a sufficiently big $m \ge 0$.

To apply this formula, we consider the system (5) without the input $u(t)$ and perturbations $d(t), \nu(t)$

$$\dot{x}(t) = (A_i - L_iC_i)x(t) + L_iy(t) \qquad (15)$$

In addition in this subsection we will assume that $\mathcal{I} = \{1, 2\}$ and that the switching is periodical and $t_{k+1} - t_k = \tau$ for some period $\tau > 0$. The state trajectories of the system between two switching times can be calculated analytically from the equation (15) of the activated subsystem

$$\begin{aligned} x(t_{k+1}) = &\; e^{\tilde{A}_{\sigma(t_k)}(t_{k+1}-t_k)} x(t_k) \\ &+ \int_{t_k}^{t_{k+1}} e^{\tilde{A}_{\sigma(t_k)}(t_{k+1}-s)} L_{\sigma(t_k)} y(s) ds \end{aligned}$$

where $\tilde{A}_{\sigma(t_k)} = A_{\sigma(t_k)} - L_{\sigma(t_k)} C_{\sigma(t_k)}$ and $\sigma(t_k)$ corresponds to the index of the activated system. After switching $2m$ times between subsystems represented by (15), the state variables will be obtained through the analytical solution

$$\begin{aligned} x(T) = &\prod_{k=0}^{2m-1} e^{\tilde{A}_{\sigma(t_k)}\tau} x(t_0) \\ &+ \sum_{k=1}^{2m} \Big[\prod_{j=k}^{2m-1} \left(e^{\tilde{A}_{\sigma(t_j)}\tau}\right) \\ &\times \int_{t_{k-1}}^{t_k} e^{\tilde{A}_{\sigma(t_{k-1})}(t_k-s)} L_{\sigma(t_{k-1})} y(s) ds \Big] \end{aligned} \tag{16}$$

where $T = 2m\tau$. To simplify, the equation (16) can be represented as follows

$$x(T) = \prod_{k=0}^{2m-1} e^{\tilde{A}_{\sigma(t_k)}\tau} x(t_0) + \Lambda(y(t)) \tag{17}$$

where

$$\begin{aligned} \Lambda(y(T)) = &\sum_{k=1}^{2m} \Big[\prod_{j=k}^{2m-1} \left(e^{\tilde{A}_{\sigma(t_j)}\tau}\right) \\ &\times \int_{t_{k-1}}^{t_k} e^{\tilde{A}_{\sigma(t_{k-1})}(t_k-s)} L_{\sigma(t_{k-1})} y(s) ds \Big] \end{aligned}$$

Using the fact that $\tilde{A}_{\sigma(t_k)} \in \{\tilde{A}_1, \tilde{A}_2\}$.

$$\begin{aligned} \prod_{k=0}^{2m-1} e^{\tilde{A}_{\sigma(t_k)}\tau} &= \left(e^{\tilde{A}_1\tau} e^{\tilde{A}_2\tau}\right)^m \\ &= \left(e^{\tilde{A}_1 \frac{T}{2m}} e^{\tilde{A}_2 \frac{T}{2m}}\right)^m \end{aligned} \tag{18}$$

At this stage by using Theorem 4 we know

$$\lim_{m\to\infty} \left(e^{\tilde{A}_1 \frac{T}{2m}} e^{\tilde{A}_2 \frac{T}{2m}}\right)^m = e^{(\tilde{A}_1+\tilde{A}_2)\frac{T}{2}} \tag{19}$$

By coupling (17) and (19), we find

$$x((l+1)T) = \left(e^{(\tilde{A}_1+\tilde{A}_2)\frac{T}{2}} - \Delta\right) x(lT) + \Lambda(y((l+1)T)) \tag{20}$$

where

$$\Delta = \left[e^{(\tilde{A}_1+\tilde{A}_2)\frac{T}{2}} - \left(e^{\tilde{A}_1 \frac{T}{2m}} e^{\tilde{A}_2 \frac{T}{2m}}\right)^m\right] \tag{21}$$

And due to Trotter result we know that the matrix Δ converges to zero by increasing m. Then the interval observer for (20) can be designed in the following form:

$$\begin{aligned} \underline{x}((l+1)T) = &e^{(\tilde{A}_1+\tilde{A}_2)\frac{T}{2}} \underline{x}(lT) - \Delta^+ \overline{x}(lT) + \Delta^- \underline{x}(lT) \\ &+ \Lambda(y((l+1)T)) \\ \overline{x}((l+1)T) = &e^{(\tilde{A}_1+\tilde{A}_2)\frac{T}{2}} \overline{x}(lT) - \Delta^+ \underline{x}(lT) + \Delta^- \overline{x}(lT) \\ &+ \Lambda(y((l+1)T)) \end{aligned} \tag{22}$$

provided that the next Assumption is valid:

Assumption 5. There exist two matrices $L_1, L_2 \in \mathbb{R}^{n\times p}$ such that the matrix $A_1 - L_1C_1 + A_2 - L_2C_2$ is Metzler.

It has been shown in (Minc, 1988; Mitkowski, 2008) that the matrix $e^{At} \geq 0$ is nonnegative if and only if $A \in \mathbb{R}^{n\times n}$ is Metzler. Thereby, Assumption 5 allows us to conclude that the matrix $e^{(\tilde{A}_1+\tilde{A}_2)\frac{T}{2}}$ is nonnegative for all $T \geq 0$.

Assumption 6. There exist two matrices $L_1, L_2 \in \mathbb{R}^{n\times p}$, a symmetric positive-definite matrix $P_\Gamma \in \mathbb{R}^{2n\times 2n}$ and a scalar $m \in \mathbb{N}$ such that the nonnegative matrix

$$\Gamma = \begin{pmatrix} e^{(\tilde{A}_1+\tilde{A}_2)\frac{T}{2}} + \Delta^- & \Delta^+ \\ \Delta^+ & e^{(\tilde{A}_1+\tilde{A}_2)\frac{T}{2}} + \Delta^- \end{pmatrix}$$

satisfies the LMI

$$\Gamma^T P_\Gamma \Gamma - P_\Gamma \prec 0.$$

Theorem 5. Let Assumptions 5 and 6 be satisfied and $x \in \mathcal{L}_\infty^n$, then in the system (20) with the interval observer (22) the relations

$$\underline{x}(lT) \leq x(lT) \leq \overline{x}(lT), \forall l \in \mathbb{Z}_+$$

are satisfied provided that $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$, and the estimates $\underline{x}(lT), \overline{x}(lT)$ are bounded.

To generalize the synthesis of the interval observer to the perturbed case the conditions imposed in Assumption 2.1 will be used. The system (15) with input disturbance can be rewritten in the following form:

$$\dot{x}(t) = (A_i - L_iC_i)x(t) + d(t) + L_i y(t) \quad i \in \mathcal{I} \tag{23}$$

The state variables, after period of time T, can be calculated with the same as previously analytical method

$$\begin{aligned} x(T) = &\prod_{k=0}^{2m-1} e^{\tilde{A}_{\sigma(t_k)}\tau} x(t_0) + \sum_{k=1}^{2m} \Big[\prod_{j=k}^{2m-1} \left(e^{\tilde{A}_{\sigma(t_j)}\tau}\right) \\ &\times \int_{t_{k-1}}^{t_k} e^{\tilde{A}_{\sigma(t_{k-1})}(t_k-s)} d(t) ds \Big] + \Lambda(y(T)) \end{aligned} \tag{24}$$

Proposition 1. Under Assumption 2.1 the inequality

$$\underline{D}_k \leq D_k \leq \overline{D}_k$$

is satisfied, where

$$\begin{aligned} \underline{D}_k &= \int_{t_{k-1}}^{t_k} \Big(\left(e^{\tilde{A}_{\sigma(t_{k-1})}(t_k-s)}\right)^+ \underline{d}(s) \\ &\quad - \left(e^{\tilde{A}_{\sigma(t_{k-1})}(t_k-s)}\right)^- \overline{d}(s) \Big) ds, \\ D_k &= \int_{t_{k-1}}^{t_k} e^{\tilde{A}_{\sigma(t_{k-1})}(t_k-s)} d(s) ds, \\ \overline{D}_k &= \int_{t_{k-1}}^{t_k} \Big(\left(e^{\tilde{A}_{\sigma(t_{k-1})}(t_k-s)}\right)^+ \overline{d}(s) \\ &\quad - \left(e^{\tilde{A}_{\sigma(t_{k-1})}(t_k-s)}\right)^- \underline{d}(s) \Big) ds. \end{aligned} \tag{25}$$

And the new-estimated bounds can be expressed as follows

$$\begin{aligned} \underline{x}((l+1)T) = &e^{(\tilde{A}_1+\tilde{A}_2)\frac{T}{2}} \underline{x}(lT) + \Delta^- \underline{x}(lT) - \Delta^+ \overline{x}(lT) \\ &+ \underline{\Pi}_{2m} + \Lambda(y((l+1)T)) \\ \overline{x}((l+1)T) = &e^{(\tilde{A}_1+\tilde{A}_2)\frac{T}{2}} \overline{x}(lT) + \Delta^- \overline{x}(lT) - \Delta^+ \underline{x}(lT) \\ &+ \overline{\Pi}_{2m} + \Lambda(y((l+1)T)) \end{aligned} \tag{26}$$

where $\underline{\Pi}_{2m}, \overline{\Pi}_{2m}$ are calculated by

$$\underline{\Pi}_k = \begin{cases} \Psi_k^+ \underline{\Pi}_{k-1} - \Psi_k^- \overline{\Pi}_{k-1} + \underline{D}_k & \text{for } k \geq 2 \\ \underline{D}_k & \text{for } k = 1 \end{cases}$$

$$\overline{\Pi}_k = \begin{cases} \Psi_k^+ \overline{\Pi}_{k-1} - \Psi_k^- \underline{\Pi}_{k-1} + \overline{D}_k & \text{for } k \geq 2 \\ \overline{D}_k & \text{for } k = 1 \end{cases}$$

where $\Psi_k = \left(e^{\tilde{A}_{\sigma(t_{k-1})}\tau}\right)$ and the quantities $\underline{D}_k, \overline{D}_k$ are defined by expressions (25).

Theorem 6. Let Assumptions 5, 6 and 2.1 be satisfied and $x \in \mathcal{L}_\infty^n$, then in the system (24) with the interval observer (26) the relations

$$\underline{x}(lT) \leq x(lT) \leq \overline{x}(lT), \forall l \in \mathbb{Z}_+$$

are satisfied provided that $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$, in addition the estimates $\underline{x}(lT)$ and $\overline{x}(lT)$ are bounded.

4. ILLUSTRATIVE EXAMPLES

For the sake of simplicity, we consider switched systems with 2 modes, i.e. $\mathcal{I} = \{1, 2\}$.

4.1 Example for the change of coordinates based interval observer

Consider system (5) with

$$A_1 = \begin{bmatrix} 0 & 1 \\ -0.35 & -1.2 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -0.8 & -1.8 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, C_1^T = C_2^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and $u(t) = 10\sin(2t)$, $d(t) = 0.1\sin(5t)E_{2,1}$, and $v(t) = 0.05\sin(6t) + 0.05\sin(t)$, which satisfy

$$-0.1 \cdot E_{2,1} = \underline{d} \leq d(t) \leq \overline{d} = 0.1 \cdot E_{2,1}, \quad |v(t)| \leq \mathcal{V} = 0.1.$$

For the observable pairs (A_i, C_i) we choose the observer gain matrices $L_1 = \begin{bmatrix} -4.43 & 1.7 \end{bmatrix}^T$, $L_2 = \begin{bmatrix} -2.75 & 1.7 \end{bmatrix}^T$.

$$D_1 = \begin{bmatrix} -1 & 0.0 \\ 0.0 & -1.9 \end{bmatrix}, D_2 = \begin{bmatrix} -1.5 & 0 \\ 0 & -2 \end{bmatrix},$$

Furthermore, the matrices for coordinates change are

$$S_1 = \begin{bmatrix} 1.9669 & -1.8877 \\ -0.3623 & 0.6607 \end{bmatrix}, S_2 = \begin{bmatrix} 2.7854 & -2.6471 \\ -1.1142 & 1.4118 \end{bmatrix}$$

And the gain matrices $M_1 = \begin{bmatrix} 0.5 & 1.8 \end{bmatrix}^T$, $M_2 = \begin{bmatrix} -2.5 & -2.5 \end{bmatrix}^T$ are chosen in a way to guarantee the positivity of reset matrices in the new coordinates. The stability of the observer is fulfilled for all $\theta \in [0.2086, +\infty)$.

The results of simulation are shown in Figure 1 for the interval observer in the new coordinates $z(t)$ by dashed lines and in Figure 2 the bounds of interval observer for the system are presented in red and blue color for upper and lower bounds respectively.

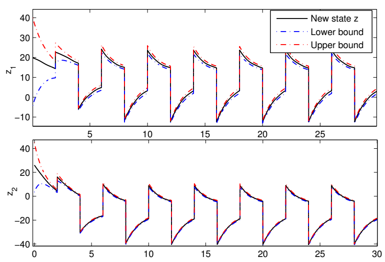

Fig. 1. Simulation results of the second approach: new state variables z.

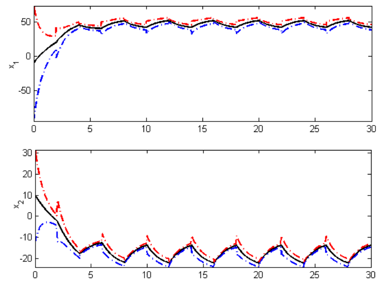

Fig. 2. Simulation results of the second approach: state variables x.

4.2 Example for the third approach

Consider system (5) with

$$A_1 = \begin{bmatrix} -10 & -4 \\ 4 & -3 \end{bmatrix}, B_1 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}, A_2 = \begin{bmatrix} -8 & 4 \\ -4 & -7 \end{bmatrix}, B_2 = \begin{bmatrix} 10 \\ 0 \end{bmatrix},$$

$$C_1 = C_2 = [0 \ 1]$$

and the model and output perturbations $d(t) = \begin{bmatrix} 0.5\sin(5t), & 0.5\sin(5t) \end{bmatrix}^T$, $v(t) = 0.5\sin(2t) + 0.5\sin(5t) + 0.2$ with $\mathcal{V} = 1.2$, $\underline{d} = \begin{bmatrix} -0.5, -0.5 \end{bmatrix}^T, \overline{d} = \begin{bmatrix} 0.5, 0.5 \end{bmatrix}^T$.

For the observer gain matrices $L_1 = \begin{bmatrix} 1 & 4 \end{bmatrix}^T$, $L_2 = \begin{bmatrix} 2 & 6 \end{bmatrix}^T$:

$$\tilde{A}_{1d} + \tilde{A}_{1o}^+ = \begin{bmatrix} -10 & 0 \\ 4 & -7 \end{bmatrix}, \tilde{A}_{1o}^- = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix};$$

$$\tilde{A}_{2d} + \tilde{A}_{2o}^+ = \begin{bmatrix} -8 & 2 \\ 0 & -13 \end{bmatrix}, \tilde{A}_{2o}^- = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}$$

are Metzler. The stability of switched interval observer

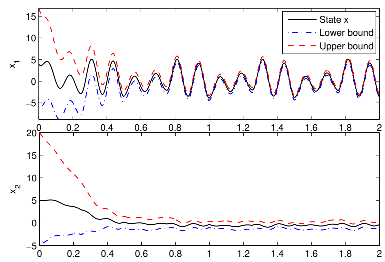

Fig. 3. Simulation results of the third approach.

(13) is verified by satisfying the LMIs (12). The simulation result is shown in Figure 3.

4.3 Example of the Trotter approximation based interval observer

Consider system (23) with

$$A_1 = \begin{bmatrix} -5 & 3 \\ -1 & -3 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix}, C_1^T = C_2^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and $d(t) = \begin{bmatrix} 2\sin(5t) + 10\sin(0.6t), 5\cos(t) + 3\sin(5t) \end{bmatrix}^T$, which satisfy $\underline{d}(t) \leq d(t) \leq \overline{d}(t)$ where

$$\underline{d}(t) = \begin{bmatrix} -2 + 10\sin(0.6t) \\ -3 + 5\cos(t) \end{bmatrix}, \ \overline{d}(t) = \begin{bmatrix} 2 + 10\sin(0.6t) \\ 3 + 5\cos(t) \end{bmatrix}.$$

For the cooperativity of estimated error bounds, the parameter $m = 2$ in the expression (21) guarantees the

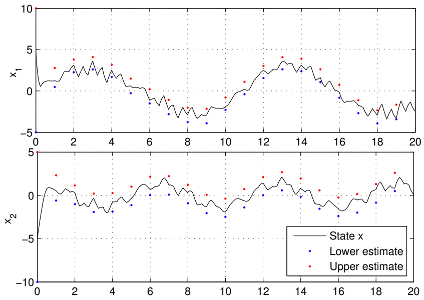

Fig. 4. Simulation results of the Trotter approximation based approach; the red (blue) points are the upper (lower) estimates of the state at discrete time instants.

nonnegativity of Δ and the Assumption 5 is fulfilled due to the matrix

$$A_1 - L_1C_1 + A_2 - L_2C_2 = \begin{bmatrix} -9.0 & 2.0 \\ 1.5 & -7.0 \end{bmatrix}$$

is Metzler. And the time of activation of each mode is $\tau = 0.25$. The stability of discrete interval observer (26) is verified by satisfying the LMI in the Assumption 6. The result of simulation is presented in Figure 4.

5. CONCLUSION

In this paper the interval observation for class of switched systems has been considered. The uncertainties are presented as unknown inputs (also called model perturbation), and output measurement noise. It is supposed that the values of these uncertainties belong to certain known intervals at any moment of time. Some sufficient conditions of positivity of estimated error dynamics have been adopted. Several approaches of designing interval observer have been proposed based on cooperative system dynamics. The three first approaches give continuous estimation, where the second one is based on a coordinates transformation approach (Raissi et al., 2012) and the third one uses the fact that any system can be presented in its internal nonnegative form (Cacace et al., 2015). The last approach is a sampled-time estimation from continuous measurements based on the Trotter approximation (Trotter, 1959). Demonstration of developed approaches have been shown through academic examples.

REFERENCES

Barbot, J., Saadaoui, H., Djemai, M., and Manamanni, N. (2007). Nonlinear observer for autonomous switching systems with jumps. *Nonlinear Analysis: Hybrid Systems*, 1(4), 537–547.

Boukhobza, T. and Hamelin, F. (2011). Observability of switching structured linear systems with unknown input. a graph-theoretic approach. *Automatica*, 47(2), 395–402.

Briat, C. (2015). Convex conditions for robust stabilization of uncertain switched systems with guaranteed minimum and mode-dependent dwell-time. *Systems & Control Letters*, 78, 63–72.

Cacace, F., Germani, A., Manes, C., et al. (2015). A new approach to design interval observers for linear systems. *IEEE Trans. Automat. Contr.*, 60(6), 1665–1670.

Degue, K., Efimov, D., and Richard, J.P. (2016). Interval observers for hybrid linear systems. In *10th IFAC Symposium on Nonlinear Control Systems (NOLCOS)*. Monterey.

Efimov, D., Fridman, L., Raissi, T., Zolghadri, A., and Seydou, R. (2012). Interval estimation for lpv systems applying high order sliding mode techniques. *Automatica*, 48(9), 2365–2371.

Efimov, D., Perruquetti, W., Raïssi, T., and Zolghadri, A. (2013a). Interval observers for time-varying discrete-time systems. *IEEE Transactions on Automatic Control.*

Efimov, D., Perruquetti, W., and Richard, J.P. (2013b). Interval estimation for uncertain systems with time-varying delays. *International Journal of Control*, 86(10), 1777–1787.

Efimov, D. and Raïssi, T. (2016). Design of interval observers for uncertain dynamical systems. *Automation and Remote Control*, 77(2), 191–225.

Efimov, D., Raïssi, T., and Zolghadri, A. (2013c). Control of nonlinear and lpv systems: interval observer-based framework. *IEEE Transactions on Automatic Control*, 58(3), 773–778.

Farina, L. and Rinaldi, S. (2000). *Positive Linear Systems: Theory and Applications*, volume 50. John Wiley & Sons.

Liberzon, D. (2003). *Switching in Systems and Control.* Springer Science & Business Media.

Liberzon, D., Tempo, R., et al. (2004). Common lyapunov functions and gradient algorithms. *IEEE Transactions on Automatic Control*, 49(6), 990–994.

Lin, H. and Antsaklis, P.J. (2009). Stability and stabilizability of switched linear systems: a survey of recent results. *IEEE Transactions on Automatic control*, 54(2), 308–322.

Mazenc, F., Dinh, T.N., and Niculescu, S.I. (2014). Interval observers for discrete-time systems. *International Journal of Robust and Nonlinear Control*, 24(17), 2867–2890.

Minc, H. (1988). *Nonnegative matrices.* Wiley.

Mitkowski, W. (2008). Dynamical properties of metzler systems. *Technical Sciences*, 56(4).

Raissi, T., Efimov, D., and Zolghadri, A. (2012). Interval state estimation for a class of nonlinear systems. *Automatic Control, IEEE Transactions on*, 57(1), 260–265.

Ríos, H., Mincarelli, D., Efimov, D., Perruquetti, W., and Davila, J. (2015). Continuous and discrete state estimation for switched lpv systems using parameter identification. *Automatica*, 62, 139–147.

Tanwani, A., Shim, H., and Liberzon, D. (2013). Observability for switched linear systems: characterization and observer design. *IEEE Transactions on Automatic Control*, 58(4), 891–904.

Trotter, H.F. (1959). On the product of semi-groups of operators. *Proceedings of the American Mathematical Society*, 10(4), 545–551.

Vu, L., Chatterjee, D., and Liberzon, D. (2007). Input-to-state stability of switched systems and switching adaptive control. *Automatica*, 43(4), 639–646.